A SHORT PROOF OF THE EXISTENCE OF THE ROST COHOMOLOGICAL INVARIANT (Cohomology Theory of Finite Groups and Related Topics)

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A SHORT PROOF OF THE EXISTENCE OF THE ROST COHOMOLOGICAL INVARIANT

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1. Introduction

Let G_k be a split linear algebraic group over a field k. The cohomological invariant $Inv^*(G_k;\mathbb{Z}/p)$ is (roughly speaking) the ring of natural maps $H^1(F;G_k)\to H^*(F;\mathbb{Z}/p)$ for finitely generated field F over k. For each simple simply connected group, Rost defined the invariant $R(G_k)\in Inv^3(G_k;\mathbb{Z}/p)$, which is nonzero whenever the corresponding complex Lie group G has p-torsion.

In this paper, we give a short proof of the existence of the Rost invariant for an algebraic closed field k in $\mathbb{C}$, by using motivic cohomology and the affirmative answer of the Bloch-Kato conjecture by Voevodsky.

2. motivic cohomology

Recall that $H^1(k;G_k)$ is the first non abelian Galois cohomology set of G_k, which represents the set of G_k-torsors over k. The cohomology invariant is defined by

$$Inv^i(G_k,\mathbb{Z}/p) = Func(H^1(F;G_k)\to H^i(F;\mathbb{Z}/p))$$

where $Func$ means natural functions for each fields F over k. (For accurate definition or properties, see the books [Ga-Me-Se], [Ga].)

Let BG_k be the classifying space ([To]) of G. Totaro proved [Ga-Me-Se] the following theorem in the letter to Serre.

Theorem 2.1. *(Totaro)* $Inv^*(G_k;\mathbb{Z}/p)\cong H^0(BG_k;H^*_{\mathbb{Z}/p})$.

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Here $H^*(X;H^{*'}_{\mathbb{Z}/p})$ is the cohomology of the Zarisky sheaf induced from the presheaf $H^*_{et}(V;\mathbb{Z}/p)$ for open subsets V of X. This sheaf cohomology is also the E_2-term

$$E_2^{*,*'} \cong H^*(BG_k;H^{*'}_{\mathbb{Z}/p}) \Longrightarrow H^*(BG_k;\mathbb{Z}/p)$$

of the coniveau spectral sequence by Bloch-Ogus [Bl-Og].

Next we recall the motivic cohomology. Let X be a smooth (quasi projective) variety over a field $k\subset\mathbb{C}$. Let $H^{*,*'}(X;\mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology defined by Voevodsky and Suslin ([Vo1-3]). Recall that the Belinson-Lichtenbaum conjecture holds if

$$H^{m,n}(X;\mathbb{Z}/p)\cong H^m_{et}(X;\mu_p^{\otimes n}) \qquad for\ all\ m\le n.$$

Recently M.Rost and V.Voevodsky ([Vo5],[Su-Jo],[Ro]) proved the Bloch-Kato conjecture. The Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture.

In this paper, we assume that k contains a primitive p-th root of unity. Then there is the isomorphism $H^m_{et}(X;\mu_p^{\otimes n})\cong H^m_{et}(X;\mathbb{Z}/p)$. Let τ be a generator of $H^{0,1}(Spec(k);\mathbb{Z}/p)\cong\mathbb{Z}/p$, so that

$$colim_i\tau^i H^{*,*'}(X;\mathbb{Z}/p)\cong H^*_{et}(X;\mathbb{Z}/p).$$

The Beilinson and Lichtenbaum conjecture also implies the exact sequences of cohomology theories

Theorem 2.2. *([Or-Vi-Vo], [Vo5]) There is the long exact sequence*

$$\to H^{m,n-1}(X;\mathbb{Z}/p)\xrightarrow{\times\tau} H^{m,n}(X;\mathbb{Z}/p)$$
$$\to H^{m-n}(X;H^n_{\mathbb{Z}/p})\to H^{m+1,n-1}(X;\mathbb{Z}/p)\xrightarrow{\times\tau}.$$

In particular, we have

Corollary 2.3. *The graded ring $grH^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p})$ is isomorphic to*

$$H^{m,n}(X;\mathbb{Z}/p)/(\tau)\oplus Ker(\tau)|H^{m+1,n-1}(X;\mathbb{Z}/p)$$

where $H^{m,n}(X;\mathbb{Z}/p)/(\tau)=H^{m,n}(X;\mathbb{Z}/p)/(\tau H^{m,n-1}(X;\mathbb{Z}/p))$.

Corollary 2.4. *The map $\times\tau: H^{m,m-1}(X;\mathbb{Z}/p)\to H^{m,m}(X;\mathbb{Z}/p)$ is injective.*

3. Lie groups

In this section, we assume that k is an algebraic closed field in $\mathbb{C}$. Let G be the complex Lie group corresponding to G_k for fields k. Suppose that G is a simple simply connected Lie group having p-torsion in $H^*(G)$, namely

$$(G,p) = \begin{cases} G_2,\ F_4,\ E_6,\ E_7,\ E_8,\ Spin_n\ (n \geq 7) & for\ p=2 \\ F_4,\ E_6,\ E_7,\ E_8 \quad for\ p=3, \\ E_8 \quad for\ p=5. \end{cases}$$

It is known that G is 2-connected and there is an element $x_3(G) \in H^3(G;\mathbb{Z}/p) \cong \mathbb{Z}/p$ with $Q_1x_3(G) \neq 0$ for the Milnor operation Q_1. Note that for each inclusion $i : G \subset G'$ for above groups, we know $i^*(x_3(G')) = x_3(G)$. Consider the classifying space BG and its cohomology. Denote by $x_4(G)$ the transgression of $x_3(G)$ in $H^4(BG;\mathbb{Z}/p)$, namely, $x_4(G)$ generates $H^4(BG;\mathbb{Z}/p) \cong \mathbb{Z}/p$ and $Q_1(x_4(G)) \neq 0$. We will write the integral lift of $x_4(G)$ also by the same letter $x_4(G)$.

Lemma 3.1. *The element $px_4(G) \in H^4(BG)_{(p)}$ is represented by the Chern class $c_2(\xi)$ of some complex representation $\xi : G \to U(N)$.*

Proof. We only need to prove for $G = Spin_n, p = 2$ and $G = E_8$ for odd primes. Because when $p = 2$, there is the inclusion $i : G \subset Spin_N$ for some N so that $i^*(x_4(Spin_N)) = x_4(G)$. For odd prime cases, there is the inlusion $i : G \subset E_8$, such that $i^*(x_4(E_8)) = x_4(G)$.

The complex representation ring is known for $N = 2n + 1$

$$R(Spin_N) \cong \mathbb{Z}[\lambda_1, ..., \lambda_{n-1}, \Delta_{\mathbb{C}}],$$

where λ_i is the i-th elementary symmetric function in variables $z_1^2 + z_1^{-2}, ..., z_n^2 + z_n^{-2}$ in $R(T) \cong \mathbb{Z}[z_1, z_1^{-1}, ..., z_n, z_n^{-1}]$ for the maximal torus T in $Spin_N$. Let T^1 be the first factor of T and $\eta : T^1 \subset Spin_N$. Then it is proved (page 1052 in [Sc-Ya]) that

$$\eta^* c_2(\lambda_1) = 4u, \quad \eta^* x_4(Spin_N) = 2u$$

where u is the generator of $H^2(BT^1;\mathbb{Z}) = \mathbb{Z}$. This implies $2x_4(Spin_N) = c_2(\lambda_1)$.

Let $\alpha : E_8 \to SO(248)$ be the adjoint representation of E_8. By the construction of the exceptional Lie group E_8 in [Ad], there exists a homomorphism β : Spin(16) $\to E_8$ such that the induced representation of $\alpha \circ \beta$ is the direct sum of λ_{16}^2 : Spin(16) $\to SO(120)$ and Δ_{16}^+ : Spin(16) $\to SO(128)$. Let T^8 be the maximal torus of Spin(16). Let T^1 be the first factor of T^8 and $\eta : T^1 \to$ Spin(16) the inclusion of T^1 into Spin(16). Then it is proved ([Ka-Ya]) that the total Chern class of the complexification of $\alpha \circ \beta \circ \eta$ is

$$1 - 120u^2 + \cdots \in \mathbb{Z}[u] \cong H^*(BT^1;\mathbb{Z}).$$

Since $120 = 2^3 \cdot 3 \cdot 5$, the Chern class $c_2(\alpha)$ represents $\gamma p x_4(E_8)$ for $p = 3, 5$ in $H^4(BE_8; \mathbb{Z}_{(p)})$, where γ is a unit in $\mathbb{Z}_{(p)}$. □

Let $t_{\mathbb{C}} : H^{*,*'}(X; \mathbb{Z}/p) \to H^*(X(\mathbb{C}); \mathbb{Z}/p)$ be the realization map ([Vo1]) for the inclusion $k \subset \mathbb{C}$. Voevodsky defines the Milnor operation Q_i also in the mod p motivic cohomology

$$Q_i : H^{*,*'}(-; \mathbb{Z}/p) \to H^{*+2p^i-1,*'+p^i-1}(-; \mathbb{Z}/p)$$

which are compatible with the usual (topological) cohomology operations by the realization map $t_{\mathbb{C}}$. For smooth X, the oparation

$$Q_i : H^{2*,*}(X; \mathbb{Z}/p) = CH^*(X)/p \to H^{2*+2p^i-1,*+p^i-1}(X; \mathbb{Z}/p) = 0$$

is zero since $2(* + p^i - 1) - (2* + 2p^i - 1) = -1 < 0$.

Theorem 3.2. *There is the nonzero element $y_3(G_k) \in Inv^3(G_k; \mathbb{Z}/p)$ which is natural for the embedding $G_k \subset G'_k$ of the groups.*

Proof. From Corollary 2.3, we see

$$Ker(\tau)|H^{4,2}(BG_k; \mathbb{Z}/p) \subset H^0(BG_k; H^3_{\mathbb{Z}/p}) \cong Inv^3(G_k; \mathbb{Z}/p).$$

Hence we only need to see the existence of a nonzero element $c \in H^{4,2}(BG_k; \mathbb{Z}/p)$ with $\tau c = 0$.

Since $Q_1(x_4(G)) \neq 0$, there is no element x in $H^{4,2}(BG_k; \mathbb{Z}/p)$ such that $t_{\mathbb{C}}(x) = x_4(G)$, while there exists in $H^{4,4}(BG_k; \mathbb{Z}/p)$ from the Beilinson-Lichtenbaum conjecture.

On the other hand, $c_2(\xi) \in CH^2(BG_k)$, in fact Chow rings have Chern classes. Since $t_{\mathbb{C}}(c_2(\xi)) = px_4(G)$, we see that $c_2(\xi)$ is an additive generator of $H^{4,2}(BG_k)_{(p)}$, so is nonzero in $H^{4,2}(BG_k; \mathbb{Z}/p)$.

Consider the element

$$\tau^2(c_2(\xi)) = px = 0 \ \in H^{4,4}(BG_k; \mathbb{Z}/p) \cong H^4(BG; \mathbb{Z}/p) \cong \mathbb{Z}/p.$$

From Corollary 2.4, the map $\times\tau : H^{4,3}(BG_k; \mathbb{Z}/p) \to H^{4,4}(BG_k; \mathbb{Z}/p)$ is injective. Hence $\tau c_2(\xi) = 0$ in $H^{4,3}(BG_k; \mathbb{Z}/p)$. □

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